On the Ending Digits of $a(n) = n^2 + n + 1$

Michael G. Kaarhus

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Abstract

This article covers two questions:

Why Do the Ending Digits of $a(n) = n^2 + n + 1$ Repeat?
Why Is the Repeating Sequence of Endings a 5 Number Palindrome? (page 3)

Introduction

I use only algebra, and consider only integers here, so "Be not afraid". $a(n) = n^2 + n + 1$ is a formula that gives the number of square roots $\geq \sqrt{n}$ and $< n + 1$. For instance, there are 7 square roots $\geq \sqrt{2}$ and $< 2 + 1$. They are $\sqrt{2}$ through $\sqrt{8}$. So $a(2) = 7$. For a text file containing the first 1001 terms of this sequence, click here. The ending digits of this sequence happen to make a repeating palindrome, $\{1, 3, 7, 7, 3, 1\}$. This was a mystery to me, so the purpose of this article is to try to figure out why.

Why Do the Ending Digits of $a(n) = n^2 + n + 1$ Repeat?

for $a(n) = n^2 + n + 1$

n	0	1	2	3	4	5	6	7	8	9	10	11
a(n)	1	3	7	13	21	31	43	57	73	91	111	133
n	12	13	14	15	16	17	18	19	20	21	22	23
a(n)	157	183	211	241	273	307	343	381	421	463	507	553
n	24	25	26	27	28	29	30	31	32	33	34	35
a(n)	601	651	703	757	813	871	931	993	1057	1123	1191	1261

Observe that the ending digit repeats every fifth term. We need a formula to get the fifth term out from any term. I will start with the first term out. Consider any 2 adjacent terms $a(n - 1)$, and

$a(n)$. I need the difference between them.

$$a(n-1) = (n-1)^2 + n - 1 + 1$$
$$a(n-1) = n^2 - n + 1 \quad \text{(This incidentally is A002061 at the OEIS)}$$
$$a(n) - a(n-1) = n^2 + n + 1 - (n^2 - n + 1)$$
$$= 2n$$

The difference is $2n$. For instance,

The difference between $a(4)$ and $a(5) = 2 \cdot 5$
The difference between $a(5)$ and $a(6) = 2 \cdot 6$
The difference between $a(6)$ and $a(7) = 2 \cdot 7$

All main sequence terms beyond the zeroth can thus be seen as the result of this nice equation:

(1) $$a(n-1) + 2n = a(n)$$

Now we want the formula for $n + 5$, because the repeating palindrome is 5 numbers long:

$$a(n+5) = (n+5)^2 + n + 5 + 1$$
$$= n^2 + 11n + 30.$$
$$a(n+5) - a(n) = n^2 + 11n + 30 - (n^2 + n + 1)$$
$$= 10n + 30$$
$$= 10(n+3) \qquad \text{Or,}$$

(2) $$a(n+5) = a(n) + 10(n+3)$$

Notice that $10(n+3)$, is a multiple of 10. Now I ask again: why does $a(n+5)$ have the same ending digit as $a(n)$? It's because, when you add $10(n+3)$ to $a(n)$, you are adding just a multiple of 10 to it; you are adding 0 to the ending digit. For instance:

$$n = 7. \quad a(7) = 57.$$
$$57 + 10(7+3) = 157, \quad \text{which is } a(12) \text{ Or,}$$
$$a(7) + 10(7+3) = a(7+5)$$

$$n = 5. \quad a(5) = 31.$$
$$31 + 10(5+3) = 111, \quad \text{which is } a(10).$$

$$n = 10. \quad a(10) = 111.$$
$$111 + 10(10+3) = 241, \quad \text{which is } a(15).$$

$$n = 21. \quad a(21) = 463.$$
$$463 + 10(21+3) = 703, \quad \text{which is } a(26).$$

Again, with the formula, $a(n + 5) = a(n) + 10(n + 3)$, you are adding only a multiple of 10 to $a(n)$, so $a(n + 5)$ will end in the same digit as $a(n)$. Simply put, since each ending digit is repeated 5 terms later, and since the initial five terms of the main sequence end {1, 3, 7, 3, 1}, each of those endings will be repeated 5 terms later, 10 terms later, 15 terms later, and so on ad infinitum.

Why Is the Repeating Sequence a Five Number Palindrome?

I might not have an ultimate answer to that. But I have made progress toward one. Our formula is $a(n) = n^2 + n + 1$. Let's begin by considering just $b(n) = n^2$, and adding terms to it. The ending digits are a direct result of the progress of the integers, and of the functions applied to them:

Beginning with the formula $b(n) = n^2$

If n ends in	0	1	2	3	4	5	6	7	8	9
b(n) ends in	0	1	4	9	6	5	6	9	4	1
b(n) + n ends in	0	2	6	2	0	0	2	6	2	0
b(n) + n + 1 ends in	1	3	7	3	1	1	3	7	3	1

Row 1: The integers end in a repeating sequence of 10 different numbers. That establishes 10 as the basic length of repetition.
Row 2: n^2 ends in a repeating sequence having 6 different numbers. Row 2 is almost palindromic, but not quite.
Row 3: $n^2 + n$ ends in a repeating sequence of just 3 different numbers. It is not only palindromic over a length of 10, it is doubly palindromic: the first 5 terms are a palindrome, as are the second 5 terms. That's where we get our repeating palindrome of length 5.
Row 4: $n^2 + n + 1$ ends like $n^2 + n$, except that all the numbers are incremented by 1. It is doubly palindromic, and gives the palindrome we seek. In fact, $n^2 + n + c$ (where c is any integer) will also give a double palindrome – having one of 10 possible different sequences of ending digits.

The above, in short, is why we have a palindrome, why it repeats, why it has 3 different numbers, why it is 5 terms long, and why it is {1, 3, 7, 3, 1}. These are all direct consequences of the progress of the integers, and of the functions applied to them.

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