A Family of Twin Prime Quads

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Abstract

This article introduces a family of quadratic equations that produce central composites (averages) of twin prime pairs. This family apparently has infinitely many members. And so it is apparent that the twin primes are infinitely many.

My previous twin primes article was about a quadratic that produces central composites:

$$y = p^2 - p - 2 \tag{1}$$

Where p is the lesser of certain twin prime pairs, y is the central composite of a larger twin prime pair, (y-1, y+1). Soon after publishing that article, I realized the p I found are part of a larger sequence of integers that produce centrals with (1). And I found many more quadratics like (1). There are apparently infinitely many of them.

One need not use twin primes in (1) to obtain centrals y. If you test all non-negative integers, you obtain a larger number of centrals. I also realized that,

$$p^2 - p - 2 \ = \ (p-1)^2 + (p-1) - 2 \tag{2}$$

You can use either the integer p with (1), or the integer $p - 1$ with the right side of (2), and obtain the *same* $(y - 1,\ y + 1)$. Here I use the integer $p - 1$, which I call x:

$$\begin{aligned} &y = x^2 + x - 2 \ \text{ yields the same centrals as} \\ &y = p^2 - p - 2, \ \text{ where } x = p - 1. \end{aligned} \tag{3}$$

For the quad in (3), all the $x \geq 4$ that yield centrals are $\equiv 1 \pmod 3$.
Plug any integer $x \equiv 1 \pmod 3$ $(x \geq 4)$ into (3):
If $(y-1,\ y+1)$ is not a twin prime pair, increment x by 3, and try again. If that's not a pair, increment x by 3, and try again. In this way, you will eventually obtain an x that yields a central, and that central will (by virtue of the quadratic) necessarily be larger than x. Integers x such that $x \equiv 1 \pmod 3$ are infinitely many. So you can, either by this method, or by some more sophisticated

one, find larger and larger $x \equiv 1 \pmod 3$ that yield even larger centrals. For instance:

$$18 = 4^2 + 4 - 2$$
$$108 = 10^2 + 10 - 2$$
$$180 = 13^2 + 13 - 2$$
$$270 = 16^2 + 16 - 2$$

The above does not prove the *Twin Prime Conjecture*, unless someone discovers something that would enable him to demonstrate that the sequence of x that produces centrals is infinite. This difficulty is typical. It might be simpler to demonstrate that the twin primes are infinitely many, by demonstrating that the quadratics that produce their centrals are infinitely many. The above quadratic (as a centrals generator) is newly-discovered, but is just one of an apparently infinite number of quads that yield central composites. At least one other such quad is known to the world. It is found in the description of one of Pierre CAMI's sequences at OEIS: Sequence A088485:

> Numbers n such that $n^2 + n - 1$ and $n^2 + n + 1$ are twin primes. (CAMI [1])

Using CAMI's equations to produce centrals instead of twin primes:

$$y = n^2 + n$$

Since we know that $y = x^2 + x - 2$ also generates centrals, perhaps there is a whole family of such quads that generate centrals. It might be

(4) $$y = x^2 + x - c, \quad \text{where } c = \text{certain integers}$$

Sure enough, that is the case. In this entire family of quads, there exist negative x that produce the same y (and the same centrals) as the non-negative x (proof below). Since the centrals produced by negative x are identical to those produced by non-negative x, I count for each quad only the centrals produced by non-negative x:

Stats for $y = x^2 + x - c$

$y =$	**Yield of centrals out of the first 1,081,000 centrals**
$x^2 + x - (-18)$	430 centrals
$x^2 + x - (-17)$	0 centrals
$x^2 + x - (-16)$	187 centrals
$x^2 + x - (-15)$	0 centrals
$x^2 + x - (-14)$	0 centrals
$x^2 + x - (-13)$	0 centrals
$x^2 + x - (-12)$	266 centrals
$x^2 + x - (-11)$	0 centrals
$x^2 + x - (-10)$	153 centrals
$x^2 + x - (-9)$	0 centrals
$x^2 + x - (-8)$	0 centrals
$x^2 + x - (-7)$	0 centrals
$x^2 + x - (-6)$	213 centrals
$x^2 + x - (-5)$	0 centrals
$x^2 + x - (-4)$	65 centrals
$x^2 + x - (-3)$	0 centrals
$x^2 + x - (-2)$	0 centrals
$x^2 + x - (-1)$	0 centrals
$x^2 + x - 0$	441 centrals
$x^2 + x - 1$	0 centrals
$x^2 + x - 2$	334 centrals
$x^2 + x - 3$	0 centrals
$x^2 + x - 4$	0 centrals
$x^2 + x - 5$	0 centrals
$x^2 + x - 6$	132 centrals
$x^2 + x - 7$	0 centrals
$x^2 + x - 8$	160 centrals
$x^2 + x - 9$	0 centrals
$x^2 + x - 10$	0 centrals
$x^2 + x - 11$	0 centrals
$x^2 + x - 12$	372 centrals
$x^2 + x - 13$	0 centrals
$x^2 + x - 14$	160 centrals (unlike $c = 8$)
$x^2 + x - 15$	0 centrals
$x^2 + x - 16$	1 central: 4 (when $x = 4$). Result tossed.
$x^2 + x - 17$	0 centrals
$x^2 + x - 18$	259 centrals

There is a family of central-producing quads here, which are identical, except for their vertical shifts (which are determined by c). If you start at 0, and add 2, then 4 alternately to each accumulating sum, ad infinitum, each accumulating sum is a correct *positive* c to subtract from $x^2 + x$. Because c is subtracted, a positive c shifts the parabola downward. And if you start at 0, and subtract 4, then 2 alternately from each accumulating sum, ad infinitum, each accumulating sum is a correct *negative* c to subtract from $x^2 + x$. A negative c shifts the parabola upward. I originally said that rather complexly:

(5) $y = x^2 + x - c$, where
$c = \{\text{all even integers } \textit{except } 2\,(2 + d), \text{ where } d = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}\}$

OEIS Editor Charles R Greathouse IV simplified the above restrictions on c to:
"c can be any even integer *not* of the form 6d + 4" (cf. A214840).
Equivalently, $\{c \,|\, c \equiv 0 \pmod 2,\ c \not\equiv 1 \pmod 3\}$

For the negative x that produce the same y values (and the same centrals) as the non-negative x, we have:

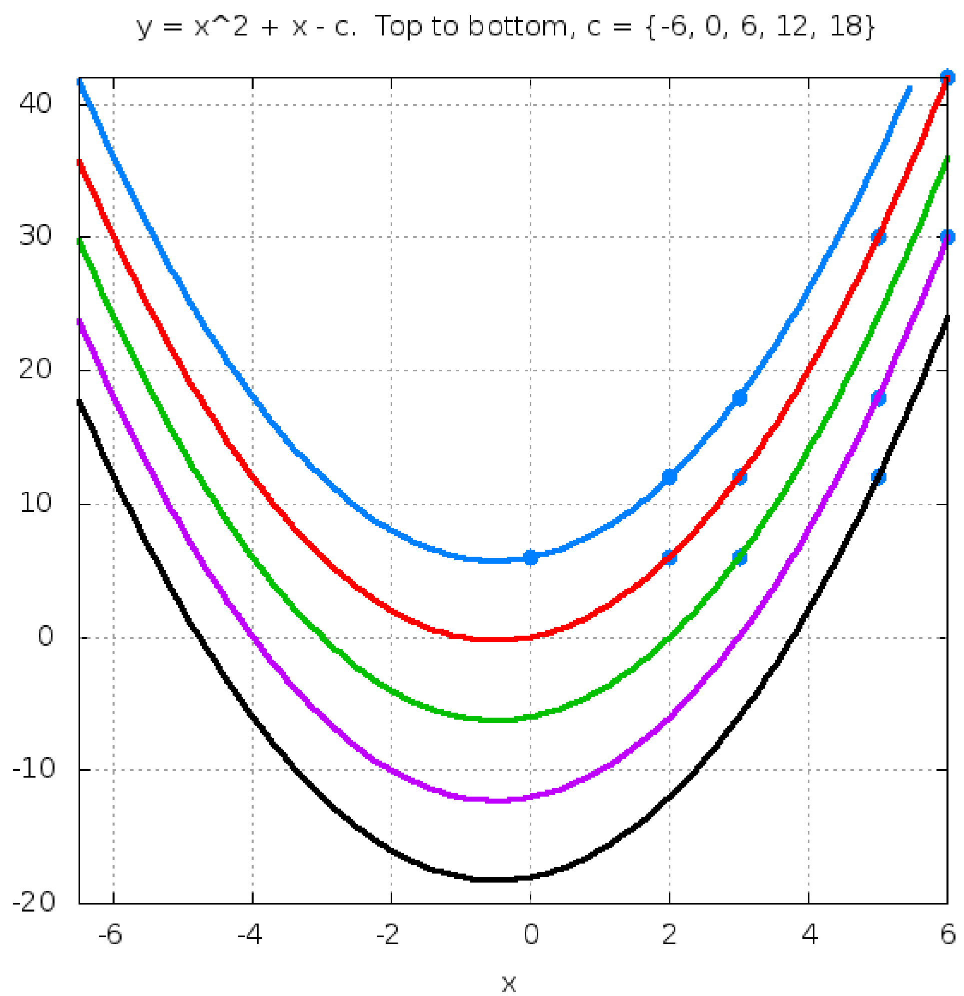
$$\begin{aligned}
f(x) &= x^2 + x - c \\
f(-x) &= (-x)^2 + (-x) - c \\
f(x-1) &= (x-1)^2 + (x-1) - c \\
(-x)^2 + (-x) - c &= (x-1)^2 + (x-1) - c \\
x^2 - x - c &= x^2 - 2x + 1 + x - 1 - c \\
x^2 - x - c &= x^2 - x - c \\
\therefore\ f(-x) &= f(x-1)
\end{aligned}$$

The negative x that produce the same centrals are, in absolute value, one more than the positive x. For instance, if $f(-9)$ produces a central, then $f(8)$ produces the same central.

All quads in this family extend upward, and are nested. To find the line containing their vertices,

$$\begin{aligned}
\frac{d}{dx}\,(x^2 + x - 2) &= 2x + 1 \\
2x + 1 &= 0 \\
x &= -\frac{1}{2}
\end{aligned}$$

Below is the domain near $x = -\frac{1}{2}$. Centrals appear as dots. Centrals from negative x are omitted:



Some quads duplicate centrals in other quads. But the sequence of centrals each quad produces is apparently unique. The family of quads in (5) includes all quads of the form

$$(x - i)\,(x + i + 1) = y, \quad \text{where } i \geq 0. \quad \text{For instance,}$$
$$(x - 0)\,(x + 1) = x^2 + x$$
$$(x - 1)\,(x + 2) = x^2 + x - 2$$
$$(x - 2)\,(x + 3) = x^2 + x - 6$$
$$(x - 3)\,(x + 4) = x^2 + x - 12, \quad \text{and so forth}$$

Each quad in the above genus of quads apparently yields a unique sequence of centrals. The genus can be described as follows:

(6) $y = x^2 + x - c$, where $x \geq 0$, and c = the product of any two consecutive positive integers.

The quads of genus (6) are simpler to describe, and might be infinitely many, but are not (for finite x) as numerous as the quads of family (5) above.

If each quad in family (5) produces a unique sequence of centrals (even if some quads duplicate some centrals generated by other quads), and if there are infinitely many of these quads (as there apparently are), then the twin primes are infinitely many.

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References

[1] P. CAMI, *Sequence A088485*, Nov. 9, 2003, OEIS Sequence A088485