Twin Prime Conjectures 1, 2 and 3

Michael G. Kaarhus

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Contents

Abstract

This article makes three new Twin Prime conjectures, develops one of them, and introduces a newly-discovered type of prime number, which I have provisionally named the *ghost primes*.

1 Introduction

Some 2300 years ago, Euclid proved that the prime numbers are infinitely many. Today, some number theorists are trying to prove or disprove the *Twin Prime Conjecture* (TPC). Its general form says simply that prime twins are infinitely many. According to math professor Daniel. A. Goldston, the first person known to make the TPC was de Polignac in 1849, who "conjectured that there will be infinitely many prime pairs with any given even difference" (Goldston [1]). I, of course, want to prove it. Here I introduce my first three twin prime conjectures. All are more specific than the general form of the TPC.

I have not learnt probability or number theory, and am not very knowledgeable in calculus. I make no attempt to prove the *Strong Conjecture*, or anything that sophisticated. My conjectures and analyses are elementary and algebraic.

Prerequisites: my readers need to understand:

- Algebra and modular arithmetic. If you don't understand modular arithmetic, you can read this: *A Primer on Modular Arithmetic*
- Twin prime pairs are primes of the form $(p,\ p+2)$. The term *twin* usually refers to just 1 integer, and *twins* means two or more. To avoid confusion I use the term *prime pair*, which means primes $(p,\ p+2)$. For instance, $(29, 31)$ is a prime pair, as is $(41, 43)$.
- At the center of each prime pair resides a *central composite* (a non-prime integer). The central composite of $(29, 31)$ is 30. The central composite of $(41, 43)$ is 42. If you find a central composite, you find a prime pair.
- (All central composites $\geq 6) \equiv 0 \pmod 6$.
- $b\#$ means b *primorial*, which is the product of all the primes up to and including b.

1.1 Variables

It is conventional to refer to a prime pair as $(p,\ p+2)$, and to declare no variable for the central composite. Here, however, I refer often to $p+2$. So I devised this convention:

$r = p + 2$

(p, r) is identical to $(p,\ p+2)$

$q =$ the central composite p+1 of the prime pair (p, r)

Hence, from the prime pair $(p,\ p+2)$, I recognize the sequence (p, q, r)

I use b exclusively for the primorial $b\#$. I use b instead of p here, because I reserve p for the first twin of a twin prime pair. The type of primorial I have in mind throughout is $b_n\#$, where $n = 0, 1, 2, 3, 4\ldots$ I omit the subscript throughout. At OEIS, this type of primorial is A002110.
I use g for a *ghost prime* (introduced below).
I use y for a g candidate.
I use z for an r candidate.

2 Conjectures 1, 2 and 3

Conjecture 1. *Between every b and $\frac{b\#}{2}$ (where b is prime) there exists at least one prime pair (p, r), such that*

$$b \le p < r < \frac{b\#}{2} \quad (b \ge 5, \quad p \ge 5).$$

If *Conjecture 1* is true, then the prime pairs are infinite, because as b and the half-primorials go to infinity, there exists at least one new and larger prime pair in each half-primorial. This conjecture seems at first to not cover the entire domain of the Naturals. However, for whatever large integer or prime pair you invoke, I can, with this conjecture invoke an even larger prime pair. For instance, if you say, "I'm thinking of the prime pair starting at 1000000007. How do you obtain one larger than that?" Simple:

There exists at least one prime pair (p, r) such that:

$$1000000009 \le p < r < \frac{1000000009\#}{2}.$$

The above is true and verified by the existence of many prime pairs, the first of which is $(1000000409, 1000000411)$. Plenty of prime pairs can be found between each b and $\frac{b\#}{2}$. *Conjecture 1* lends some intuitive form, and some generous bounds to the *General Conjecture*. I have found no instance in which *Conjecture 1* is false, but I have not proven it true.

I tried thinking of $\frac{b\#}{2}$ as a sum, rather than a product:

$$\frac{b\#}{2} = \text{ (some even integer) } + r, \text{ where } r \text{ is the second twin of a prime pair.}$$

When I considered the even integer, I discovered something odd:

$$\text{The even integer } = 4g, \text{ where } g \text{ is prime.}$$

I call g a *ghost prime* of the prime pair (p, r). I call it that because it is associated with the prime pair, but not in an apparent or easily detectable way.

The empirical evidence suggests that each $r \geq 7$ sums with at least one $4g$ to equal some primorial over 2. I find no r that does not sum with some $4g$ to equal some primorial over 2. Their relation is as follows:

(1) $\dfrac{b\#}{2} = 4g + r \quad (\text{where } b \leq p < r < \dfrac{b\#}{2}, \quad b \geq 5, \quad p \geq 5, \quad \text{and } g \text{ and } b \text{ are prime}).$

Since $r = p + 2$, the above can be rearranged to:

$$\frac{b\#}{2} = 4g + p + 2$$
$$p = \frac{b\#}{2} - 4g - 2, \text{ and}$$
$$r = \frac{b\#}{2} - 4g$$

Definition 1. *I define $r, g,$ and b to be* linked *to each other, if* $r = \frac{b\#}{2} - 4g$.

For instance, These statements are equivalent:

g and r are linked. *g and $q + 1$ are linked.* *g and $p + 2$ are linked.*

Definition 2. *I define $p, g,$ and b to be* linked *to each other, if* $p = \frac{b\#}{2} - 4g - 2$.

Definition 3. *I define a set of linked $\{r, g, b\}$ or $\{p, g, b\}$ to be a* linked set.

Definition 4. *I define a linked set to be* unique *if no other linked set is identical to it.*

Conjecture 2. *For each prime pair (p, r) $(p \geq 5)$, the twin prime p is in at least one unique linked set $\{p, g, b\}$, and the twin prime r is in at least one unique linked set $\{r, g, b\}$, where the same g and the same b are distributed to both linked sets, and where*

$$\textit{The prime pair } (p,\ r) = \left(\frac{b\#}{2} - 4g - 2, \quad \frac{b\#}{2} - 4g\right) \quad (\textit{the same restrictions as in (1)})$$

Conjecture 3. *The g are infinitely many, therefore, the p, the r, and the (p, r) are infinitely many.*

For some reason, it remains difficult, even for professional mathematicians, to prove that the sequence of the p, or the q, or the r, or the (p, r) is infinite. As an alternative, then, it might eventually be shown that the (p, r) are infinitely many, by showing that the (p, r) are linked to the g, and that the g are infinitely many.

I have not proven that the sequence of the g is infinite. So I set aside *Conjecture 3* for now. The rest of this paper will provide empirical evidence for *Conjecture 2*. I will also make bailing-wire analyses on the nature of the g.

3 Evidence That Each r Is In a Unique Linked Set {r, g, b}

3.1 123 Ghost Primes Linked to the 75 Smallest r > 5

For all the prime pairs I have checked, I find that each r is in a unique linked set $\{r, g, b\}$. Listed below are 123 linked sets from the first 75 smallest $p \geq 5$ – from $(5, 7)$ to $(2711, 2713)$. The [n] counts the prime pairs ≥ 5. The linking of the g to the (p, r) is not 1-to-1; some (p, r) are linked to more than one g. To list one of those, I use a lower case letter instead of incrementing [n]. I list *all* the g linked to each (p, r), only up to the pair $(281, 283)$. To download a text file containing the numbers below with full-length g, *click here.*

Linked Sets in the First 75 Prime Pairs ≥ 5

$[n]$	(p, r)	g	$b\#$
[1]	$(5, 7)$	2	5#
[2]	$(11, 13)$	23	7#
[3]	$(17, 19)$	63809	17#
[4]	$(29, 31)$	281	11#
[5]	$(41, 43)$	63803	17#
[6]	$(59, 61)$	11	7#
[b]	$(59, 61)$	$\approx 7.68612228235614 \cdot 10^{16}$	47#
[7]	$(71, 73)$	1212443	19#
[b]	$(71, 73)$	$\approx 4.07364480964875 \cdot 10^{18}$	53#
[8]	$(101, 103)$	263	11#
[b]	$(101, 103)$	$\approx 1.46610476699258 \cdot 10^{22}$	61#
[9]	$(107, 109)$	$\approx 4.02205595917584 \cdot 10^{29}$	79#
[10]	$(137, 139)$	3719	13#
[b]	$(137, 139)$	808711619	29#
[c]	$(137, 139)$	$\approx 3.95125683005220 \cdot 10^{45}$	113#
[11]	$(149, 151)$	251	11#
[12]	$(179, 181)$	$\approx 7.68612228235613 \cdot 10^{16}$	47#
[13]	$(191, 193)$	38031282940853	41#
[b]	$(191, 193)$	$\approx 3.72744909724140 \cdot 10^{69}$	179#
[14]	$(197, 199)$	239	11#
[b]	$(197, 199)$	27886559	23#
[c]	$(197, 199)$	$\approx 3.33830644611594 \cdot 10^{31}$	83#
[d]	$(197, 199)$	$\approx 9.00597720377625 \cdot 10^{51}$	137#
[15]	$(227, 229)$	$\approx 6.74668286600693 \cdot 10^{71}$	181#
[16]	$(239, 241)$	1212401	19#
[b]	$(239, 241)$	$\approx 3.72744909724140 \cdot 10^{69}$	179#
[17]	$(269, 271)$	27886541	23#
[b]	$(269, 271)$	$\approx 2.97109273704319 \cdot 10^{33}$	89#

Linked Sets in the First 75 Prime Pairs ≥ 5 (cont.)

$[n]$	(p, r)	g	$b\#$
$[c]$	$(269, 271)$	$\approx 3.20797014699374 \cdot 10^{41}$	107#
$[d]$	$(269, 271)$	$\approx 2.81649418739788 \cdot 10^{58}$	151#
$[e]$	$(269, 271)$	$\approx 2.06455588073781 \cdot 10^{102}$	257#
[18]	$(281, 283)$	63743	17#
$[b]$	$(281, 283)$	$\approx 1.63534516645868 \cdot 10^{15}$	43#
$[c]$	$(281, 283)$	$\approx 2.91077955448121 \cdot 10^{37}$	101#
$[d]$	$(281, 283)$	$\approx 3.95125683005220 \cdot 10^{45}$	113#
$[e]$	$(281, 283)$	$\approx 2.48702970489613 \cdot 10^{76}$	193#
$[f]$	$(281, 283)$	$\approx 5.55654523159023 \cdot 10^{92}$	233#
Below are listed all the $g \leq \frac{109\#}{2} - r$, or the first g:			
[19]	$(311, 313)$	927592266773	37#
[20]	$(347, 349)$	$\approx 7.68612228235613 \cdot 10^{16}$	47#
$[b]$	$(347, 349)$	$\approx 9.82290193885033 \cdot 10^{23}$	67#
[21]	$(419, 421)$	25070061161	31#
[22]	$(431, 433)$	927592266743	37#
[23]	$(461, 463)$	173	11#
$[b]$	$(461, 463)$	$\approx 1.46610476699258 \cdot 10^{22}$	61#
$[c]$	$(461, 463)$	$\approx 2.88195995493189 \cdot 10^{35}$	97#
$[d]$	$(461, 463)$	$\approx 3.20797014699374 \cdot 10^{41}$	107#
[24]	$(521, 523)$	3623	13#
[25]	$(569, 571)$	63671	17#
$[b]$	$(569, 571)$	$\approx 6.97426037658373 \cdot 10^{25}$	71#
[26]	$(599, 601)$	$\approx 1.86522793867409 \cdot 10^{56}$	149#
[27]	$(617, 619)$	63659	17#
$[b]$	$(617, 619)$	808711499	29#
$[c]$	$(617, 619)$	$\approx 2.40345043769276 \cdot 10^{20}$	59#
[28]	$(641, 643)$	3593	13#
[29]	$(659, 661)$	$\approx 9.82290193885033 \cdot 10^{23}$	67#
[30]	$(809, 811)$	63611	17#
$[b]$	$(809, 811)$	$\approx 3.49668746022318 \cdot 10^{43}$	109#
[31]	$(821, 823)$	83	11#
[32]	$(827, 829)$	$\approx 1.52052822216147 \cdot 10^{147}$	367#
[33]	$(857, 859)$	3539	13#
$[b]$	$(857, 859)$	63599	17#
[34]	$(881, 883)$	3533	13#
$[b]$	$(881, 883)$	808711433	29#
[35]	$(1019, 1021)$	$\approx 9.82290193885033 \cdot 10^{23}$	67#
[36]	$(1031, 1033)$	38031282940643	41#

Linked Sets in the First 75 Prime Pairs ≥ 5 (cont.)

Below are listed all the $g \leq \frac{109\#}{2} - r$, or the first g:			
$[n]$	(p, r)	g	$b\#$
$[b]$	$(1031, 1033)$	$\approx 4.07364480964875 \cdot 10^{18}$	53#
[37]	$(1049, 1051)$	3491	13#
$[b]$	$(1049, 1051)$	808711391	29#
[38]	$(1061, 1063)$	23	11#
$[b]$	$(1061, 1063)$	27886343	23#
$[c]$	$(1061, 1063)$	$\approx 2.88195995493189 \cdot 10^{35}$	97#
[39]	$(1091, 1093)$	$\approx 9.82290193885033 \cdot 10^{23}$	67#
[40]	$(1151, 1153)$	1212173	19#
[41]	$(1229, 1231)$	27886301	23#
$[b]$	$(1229, 1231)$	$\approx 2.88195995493189 \cdot 10^{35}$	97#
[42]	$(1277, 1279)$	$\approx 4.89944851864539 \cdot 10^{78}$	197#
[43]	$(1289, 1291)$	$\approx 3.49668746022318 \cdot 10^{43}$	109#
[44]	$(1301, 1303)$	27886283	23#
$[b]$	$(1301, 1303)$	$\approx 2.88195995493189 \cdot 10^{35}$	97#
[45]	$(1319, 1321)$	$\approx 7.20769027496993 \cdot 10^{62}$	163#
[46]	$(1427, 1429)$	$\approx 2.43917449943822 \cdot 10^{124}$	311#
[47]	$(1451, 1453)$	25070060903	31#
$[b]$	$(1451, 1453)$	$\approx 2.99810294111565 \cdot 10^{39}$	103#
[48]	$(1481, 1483)$	63443	17#
$[b]$	$(1481, 1483)$	808711283	29#
$[c]$	$(1481, 1483)$	$\approx 5.09121007490612 \cdot 10^{27}$	73#
[59]	$(1487, 1489)$	$\approx 4.07364480964875 \cdot 10^{18}$	53#
[50]	$(1607, 1609)$	927592266449	37#
$[b]$	$(1607, 1609)$	38031282940499	41#
$[c]$	$(1607, 1609)$	$\approx 4.07364480964875 \cdot 10^{18}$	53#
[51]	$(1619, 1621)$	$\approx 1.25183083132489 \cdot 10^{54}$	139#
[52]	$(1667, 1669)$	25070060849	31#
$[b]$	$(1667, 1669)$	$\approx 4.02205595917584 \cdot 10^{29}$	79#
[53]	$(1697, 1699)$	3329	13#
$[b]$	$(1697, 1699)$	63389	17#
$[c]$	$(1697, 1699)$	$\approx 5.09121007490612 \cdot 10^{27}$	73#
[54]	$(1721, 1723)$	3323	13#
$[b]$	$(1721, 1723)$	808711223	29#
$[c]$	$(1721, 1723)$	$\approx 1.63534516645832 \cdot 10^{15}$	43#
$[d]$	$(1721, 1723)$	$\approx 6.97426037658373 \cdot 10^{25}$	71#
$[e]$	$(1721, 1723)$	$\approx 2.91077955448121 \cdot 10^{37}$	101#
$[f]$	$(1721, 1723)$	$\approx 3.49668746022318 \cdot 10^{43}$	109#

Linked Sets in the First 75 Prime Pairs ≥ 5 (cont.)

For the pairs below, only the first g is listed:			
$[n]$	(p, r)	g	$b\#$
[55]	$(1787, 1789)$	$\approx 4.02205595917584 \cdot 10^{29}$	79#
[56]	$(1871, 1873)$	$\approx 5.99411350659951 \cdot 10^{183}$	449#
[57]	$(1877, 1879)$	$\approx 2.88195995493189 \cdot 10^{35}$	97#
[58]	$(1931, 1933)$	$\approx 9.82290193885033 \cdot 10^{23}$	67#
[59]	$(1949, 1951)$	$\approx 2.88195995493189 \cdot 10^{35}$	97#
[60]	$(1997, 1999)$	$\approx 3.33830644611594 \cdot 10^{31}$	83#
[61]	$(2027, 2029)$	$\approx 5.01809617416629 \cdot 10^{47}$	127#
[62]	$(2081, 2083)$	808711133	29#
[63]	$(2087, 2089)$	38031282940379	41#
[64]	$(2111, 2113)$	1211933	19#
[65]	$(2129, 2131)$	3221	13#
[66]	$(2141, 2143)$	$\approx 3.33830644611594 \cdot 10^{31}$	83#
[67]	$(2237, 2239)$	$\approx 3.20797014699374 \cdot 10^{41}$	107#
[68]	$(2267, 2269)$	$\approx 7.68612228235608 \cdot 10^{16}$	47#
[69]	$(2309, 2311)$	$\approx 1.46610476699258 \cdot 10^{22}$	61#
[70]	$(2381, 2383)$	$\approx 1.20368427591997 \cdot 10^{65}$	167#
[71]	$(2549, 2551)$	$\approx 3.33830644611594 \cdot 10^{31}$	83#
[72]	$(2591, 2593)$	1211813	19#
[73]	$(2657, 2659)$	3089	13#
[74]	$(2697, 2689)$	1211789	19#
[75]	$(2711, 2713)$	$\approx 1.86522793867409 \cdot 10^{56}$	149#

3.2 g and b Linked to Eight Larger r

I also selected at random a sequence of eight larger prime pairs, and calculated the first g for them. These eight pairs are from (433000571, 433000573) to (433002671, 433002673):

Linked Sets for 8 larger r

For the r below, only the first g is listed:		
r	g	$b\#$
433000573	$\approx 7.68612227153112 \cdot 10^{16}$	47#
433000651	$\approx 5.09121007490612 \cdot 10^{27}$	73#
433001071	$\approx 3.33830644611594 \cdot 10^{31}$	83#
433001209	$\approx 4.07364480954050 \cdot 10^{18}$	53#
433001551	$\approx 3.20797014699374 \cdot 10^{41}$	107#
433001689	$\approx 3.72744909724140 \cdot 10^{69}$	179#
433001971	700461161	29#
433002673	$\approx 3.72744909724140 \cdot 10^{69}$	179#

To download a text file containing the above with full-length g, click here.

3.3 The Smallest Ghost Primes

Of particular interest to number theorists is the beginning of the sequence of the g, that is, the smallest g. If the sum of the reciprocals of the g diverges, then the sequence of the g is infinite. However, you cannot determine convergence or divergence without the beginning of the sequence (as the beginning reciprocals are the largest). The above lists go from smaller to larger r, but the linked g do not increase as the r increase. To the contrary, some of the g decrease as the r increase: if you keep $\frac{b\#}{2}$ constant, and increase r, g must necessarily decrease. Some of the g listed above are among the smallest g, but many of the smallest g are not listed above.

Previously, I published here a list of what I thought were the 284 smallest g. I then submitted them to *OEIS (The Online Encyclopedia of Integer Sequences).* Charles Greathouse, (OEIS Editor-In-Chief) detected numerous errors and omissions in the numbers I submitted. So he wrote a *PARI* script (Greathouse [2]) to generate small g, and made a correct table of the 204 smallest g (Greathouse [3]). They can be found at:

PARI script by Charles Greathouse
Table of n, a(n) for n = 1..204 by Charles Greathouse

My r-g-b relation (Kaarhus [4]) is good, and *OEIS* found no exception to *Conjecture 2*:

my sequence (A218046) (corrected by Charles Greathouse) at OEIS

The first 8 elements of the sequence of the g are:

$$\{2, 11, 23, 83, 113, 131, 173, 191\}$$

Smallest g Christmas Tree

p, r	g	$b\#$
$(5, 7)$	2	5#
$(59, 61)$	11	7#
$(11, 13)$	23	7#
$(821, 823)$	83	11#
$(14561, 14563)$	113	13#
$(254729, 254731)$	131	17#
$(461, 463)$	173	11#
$(14249, 14251)$	191	13#

Convergence neither proves nor disproves that a sequence is finite. Divergence proves the sequence infinite. But the reciprocal sums of many infinite sequences converge. For instance, the reciprocal sums of the squares converge, even though the squares are infinitely many.

4 Bailing-wire Analyses

4.1 Theorem: g+1 is congruent to 0 (mod 6)

I observe that each $(g+1) \equiv 0 \pmod 6$. Each $g+1$ differs from each q, and from each $\frac{b\#}{2}+3$ by different multiples of 6. I observe also that each g differs from each other g by some integral multiple (≥ 1) of 6. I prove this with a sieve. But first, some preliminary stuff.

This step is simple but essential. I did not see this for a long time: Start with

$$\frac{b\#}{2} = 4g+1+q, \quad \text{Add 3 to both sides:}$$
$$\frac{b\#}{2}+3 = 4g+4+q$$
$$\frac{b\#}{2}+3 = 4(g+1)+q$$

That makes (the left side) $\equiv 0 \pmod 6$, and produces a g+1 element. I want one, because g is a prime such that $(g+1) \bmod 6 = 0$. I will demonstrate that here. Warning! This will be neither elegant nor concise!

$$\text{Consider } \frac{b\#}{2} = 4g+1+q.$$
$$b\# \bmod 6 = 0, \text{ and } q \bmod 6 = 0 \text{ (well-known to number theorists).}$$
$$\frac{b\#}{2} \bmod 3 = 0, \text{ and } q \bmod 3 = 0, \quad \text{which means}$$
$$4g+1 \bmod 3 = 0 \text{ (since (all the other terms) mod } 3 = 0\text{). However,}$$
$$\frac{b\#}{2} \bmod 6 \neq 0 \text{ and } 4g+1 \bmod 6 \neq 0. \text{ Why is that?}$$

$\frac{b\#}{2}$ is not congruent to 0 (mod 6), because $\frac{b\#}{2}$ has no factor of 2, or of 6. It has no factor of 2, because when you divide any primorial by 2, you remove its one and only factor of 2. $\frac{b\#}{2}$ has no factor of 6, because 6 is not prime, and you can't make a product of 6 from primes unless you have a 2.

Why is $(4g+1) \bmod 6 \neq 0$? g is by definition prime, which means:

g ends in one of these digits: $\{1, 3, 7, 9\}$.
$4g$ then must end $\{4, 2, 8, 6\}$. And
$4g+1$ must end $\{5, 3, 9, 7\}$. However,
(any integer) $\equiv 0 \pmod 6$ must end $\{0, 2, 4, 6, 8\}$.
In short, $4g+1 \bmod 6 \neq 0$ because $4g+1$ never ends $\{0, 2, 4, 6, 8\}$.

So far, I conclude that:

$$\begin{aligned}
&\frac{b\#}{2} \bmod 3 = 0,\\
&\frac{b\#}{2} \bmod 6 \neq 0.\\
&(4g+1) \bmod 3 = 0, \text{ and}\\
&(4g+1) \bmod 6 \neq 0.
\end{aligned}$$

Now consider $\frac{b\#}{2} + 3 = 4(g+1) + q$. It can be shown that, for any integer s,

$$\text{if } s \bmod 3 = 0, \text{ and } s \bmod 6 \neq 0, \text{ then } (s+3) \bmod 6 = 0.$$

For instance,

$$\begin{aligned}
&9 \bmod 6 \neq 0, \text{ but } (9+3) \bmod 6 = 0.\\
&15 \bmod 6 \neq 0, \text{ but } (15+3) \bmod 6 = 0. \text{ And so forth. And:}\\
&4g+1+3 = 4(g+1). \text{ Therefore,}\\
&\left(\frac{b\#}{2} + 3\right) \bmod 6 = 0, \text{ and}\\
&4(g+1) \bmod 6 = 0.
\end{aligned}$$

OK. That proves $4(g+1) \equiv 0 \pmod 6$. What about simply

$$(g+1) \equiv 0 \pmod 6\text{? How do I prove that?}$$

I prove that by using a sieve:

Theorem 1. *(g+1)* $\equiv 0 \pmod 6$

Proof. Make a vertical list or column of all the integral multiples of 6 from 6 to 120. Call these integers s, such that $s \bmod 6 = 0$. The column has 20 integers. $4(g+1)$ is an *s-like* integer, in that $4(g+1) \bmod 6 = 0$ (as was already shown). Alongside the first column, make a second column whose values are $s/4$. Ten of them will not be integers. Cross them out, leaving only the 10 integral $s/4$ values. For each integer remaining in the second column, observe that the adjacent s value in the first column is a multiple of 12. Since

$$\begin{aligned}
&\frac{4(g+1)}{4} \text{ is an integer, it must be that}\\
&4(g+1) \bmod 12 = 0. \text{ That is,}\\
&4(g+1) \text{ is analogous to only those } s \text{ wherein } s \bmod 12 = 0.
\end{aligned}$$

Also, g is prime, so g is odd, which means $g+1$ is even, which means $4(g+1)$ is even. Good. Cross out all the odd integers in the second column. Only five integers remain in the second column, and they are all even. For each of them, observe that the adjacent s value in the first column is a multiple of 24. Since

$\frac{4(g+1)}{4}$ is an even integer, it must be that

$4(g+1) \bmod 24 = 0$. That is,

$4(g+1)$ is analogous to only those s wherein $s \bmod 24 = 0$.

Modular Sieve

s	$s/4$	$s/4 =$ even integer?	$s \equiv 0 \pmod{24}$?	$s/4 \equiv 0 \pmod{6}$?
6	1.5	n	n	n
12	3	n	n	n
18	4.5	n	n	n
24	6	yes	yes	yes
30	7.5	n	n	n
36	9	n	n	n
42	10.5	n	n	n
48	12	yes	yes	yes
54	13.5	n	n	n
60	15	n	n	n
66	16.5	n	n	n
72	18	yes	yes	yes
78	19.5	n	n	n
84	21	n	n	n
90	22.5	n	n	n
96	24	yes	yes	yes
102	25.5	n	n	n
108	27	n	n	n
114	28.5	n	n	n
120	30	yes	yes	yes

$$\text{If } s \bmod 24 = 0, \text{ then } \frac{s}{4} \bmod 6 = 0.$$

From the above, I see that:

$$4(g+1) \bmod 24 = 0 \text{ (just like } s \bmod 24\text{). Therefore,}$$

$$\left(\frac{4(g+1)}{4}\right) \bmod 6 = 0 \left(\text{just like } \frac{s}{4} \bmod 6\right). \text{ And since}$$

$$\frac{4(g+1)}{4} = g+1,$$

$$(g+1) \bmod 6 = 0.$$

These facts can be integrated into a script that searches for g. This also means,

$$g \equiv 5 \pmod 6 \text{ (This is a p-like attribute of g)}$$

Each g differs from each other g by an integral multiple of 6, and
Each g differs from each p by an integral multiple of 6:

$$(g_2 - g_1) \equiv 0 \pmod 6 \quad (g > 2).$$
$$|g - p| \equiv 0 \pmod 6 \quad (g > 2).$$

The smallest two g that differ by only 6 are (233, 239).

4.2 Theorem: Each linked g and r end in the same digit.

I also find that all g greater than 2 must end in the *same* digit as the r to which it is linked. The r are prime, so they must end in an odd digit other than 5. Specifically, the r can never end in 7, because if they could, then the p could end in 5 (but the p have to be prime). So, the r must end $\{1, 3, \text{ or } 9\}$. Now let's figure out how the g must end.

Theorem 2. *Each g ends in the* same *1, 3, or 9 as the r to which it is linked.*

Proof. The evaluations of both sides must end in the same digit:

$$\begin{aligned} &\frac{b\#}{2} = 4g + r, \text{ and} \\ &\frac{b\#}{2}\ (b \geq 5) \text{ always ends in 5, which means} \\ &4g + r \text{ must end in 5. However, } r \text{ ends } \{1, 3, 9\}. \end{aligned} \tag{2}$$

For a *very* simple proof of (2), click here: C.
From the above, the following deductions can be made:
There are three possible endings for r, so there are three cases to consider for $4g$ and g:

If r ends in 1, $4g$ must end in 4, and g ends in 1 (because $1 \cdot 4$ ends in 4).
If r ends in 3, $4g$ must end in 2, and g ends in 3 (because $3 \cdot 4$ ends in 2).
If r ends in 9, $4g$ must end in 6, and g ends in 9 (because $9 \cdot 4$ ends in 6).

g cannot end in 7, because $4g$ must end in $\{4, 2, 6\}$, but $4 \cdot 7$ ends in 8. Therefore, each g ends in the *same* 1, 3, or 9 as the r to which it is linked. □

As a result of the fact that each linked g and r end in the same digit,

$$\text{If } g \text{ and } r \text{ are linked, then } |g - r| \equiv 0 \pmod{10}$$

These facts can also make it easier to search for g.

4.3 The g are somewhat p-like, and somewhat r-like

Since

$$(g+1) \bmod 6 = 0,$$
$$(g-1) \bmod 6 = 4.$$

That is apparent by considering any integer congruent to 0 (mod 6), and subtracting 2 from it. For example,

$$(12-2) \bmod 6 = 4$$
$$(18-2) \bmod 6 = 4, \text{ and so on.}$$

This means $g-1$ is never a q ($g-1$ is never in the *equivalence class* of any q). Thus it is true that g is never the larger twin of a prime pair. However, $g+1$ can be a q. The g ending in $\{1,9\}$ can be the smaller twin of a prime pair. But the g ending in 3 cannot. So, even though each g ends with the same $\{1,3,9\}$ as the r to which it is linked, the g are not otherwise like the r.

4.4 Theorem: If b = r, then y has a factor of r

For any one $\frac{b\#}{2}$, only one g is linked to some r_1. This is a consequence of simple arithmetic: if you change r, you must also find a different $4g$ to sum to the same $\frac{b\#}{2}$. For some *different* $\frac{b\#}{2}$, a *different* g can be linked to the *same* r_1. So, when searching for g linked to a given r, how far do you have to keep incrementing the primorial, until you are sure there are no more g linked to it?

Theorem 3. *Where*

$$y = \frac{b\#}{8} - \frac{r}{4}$$

If $b \geq r$, and y is integral, then y has a factor of r. You need increment the primorial no further than p#. The number of different g linked any one r is finite.

Proof. Multiply the second term by 2/2:

$$y = \frac{b\#}{8} - \frac{2r}{8}$$

If $b \geq r$, then r is necessarily divides $b\#$. Expand the primorial and rewrite:

$$y = \frac{(2 \cdot 3 \cdot 5 \cdot \ldots \cdot p \cdot r) - (2 \cdot r)}{8} \tag{3}$$

In some cases, the y in (3) is an integer. But whether or not y is an integer, r divides y. That's because both terms in the numerator have a factor of r. To check, divide y by r:

$$\frac{y}{r} = \frac{(2 \cdot 3 \cdot 5 \cdot \ldots \cdot p \cdot r) - (2 \cdot r)}{8 \cdot r}$$

$$\frac{y}{r} = \frac{(2 \cdot 3 \cdot 5 \cdot \ldots \cdot p) - 2}{8}$$
$$\frac{y}{r} = \frac{p\# - 2}{8}$$

Therefore, if r is a factor of $b\#$, then, if $\frac{b\#}{8} - \frac{r}{4}$ is integral, it has a factor of r. To find more g you can increment the primorial only until $b = p$; The number of different g linked any one r is finite. □

Corollary 1. *Regarding the above, y is not an integer for every $r\#$, but, whether or not y is an integer, y has the same fractional part as y/r:*

$$\frac{r\# - 2r}{8} \quad \textit{and} \quad \frac{p\# - 2}{8} \quad \textit{have the same fractional parts}$$

Dividing y by r does not change the fractional part of y. If y is an integer, dividing y by r does not change its fractional part of 0.

The above proof and corollary might be more easily seen by example.

Example 1. *Let's say you want to find ghost primes linked to the prime pair $(17, 19)$. The central composite q is 18, which is less than all half-primorials $\geq \frac{7\#}{2}$. Of those, one needs to check only*

$$\left\{\frac{7\#}{2}, \frac{11\#}{2}, \frac{13\#}{2}, \frac{17\#}{2}\right\}$$

That's because of the restriction that $b \leq p$. But who follows rules? Let's see what happens if we let $b = 19$, and search for g in $\frac{19\#}{2}$. Will we find one there?

$$\frac{19\#}{8} - \frac{19}{4} = 71316.5 \textit{ (not integral). However, 19 divides 71316.5:}$$

$$\frac{71316.5}{19} = 3753.5$$

Dividing 71316.5 by 19 causes no change in the its fractional part.

Let's try $b = 23$, and search for g in $\frac{23\#}{2}$. Will we find one there? Remember, $r = 19$, so b (which is 23) has gone beyond r:

$$\textit{Let } y = \frac{23\#}{8} - \frac{2 \cdot 19}{8} \textit{ be represented as}$$

(4) $$y = \frac{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23) - (2 \cdot 19)}{8}$$

Whether or not the y in (4) is an integer, 19 necessarily divides y. To check, divide y by 19:

$$\frac{y}{19} = \frac{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23) - (2 \cdot 19)}{8 \cdot 19}$$

$$\frac{y}{19} = \frac{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23) - 2}{8}$$
$$\frac{y}{19} = \frac{11741730 - 2}{8}$$
$$\frac{y}{19} = 1467716$$

For the prime pair $(17, 19)$*, if* r *is a factor of* $b\#$*, then all integral* y *will have a factor of 19, no matter how much larger than* 19 *you make* b*. That's because all primorials* $\geq 19\#$ *have a factor of 19, as does* $2 \cdot 19$*. That is, both terms of the numerator have a factor of 19. Keep* $b < 19$*, and* y *cannot have a factor of 19:*

$$y = \frac{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17) - (2 \cdot 19)}{8} \tag{5}$$

19 cannot divide the y *in (5). To check, divide* y *by 19:*

$$\frac{y}{19} = \frac{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17) - (2 \cdot 19)}{8 \cdot 19}$$
$$\frac{y}{19} = \frac{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17)}{8 \cdot 19} - \frac{2}{8}$$
$$\frac{y}{19} = 3358.618421\ldots - \frac{1}{4}$$
$$\frac{y}{19} = 3358.368421\ldots$$

In the above 19 does not divide y*. Neither does anything else, except 1 and* y*:*

$$y = \frac{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17) - (2 \cdot 19)}{8}$$
$$y = \frac{510510 - 38}{8}$$
$$y = \frac{510472}{8}$$
$$y = 63809 \text{ (which is a ghost prime)}$$

So, to find ghost primes for $(17, 19)$*, if you let* b *exceed 17, every integral* y *will have a factor of 19, and every non-integral* y *is by definition not prime. You will therefore find no* g *for* $(17, 19)$ *if* $b > 17$*.*

In general, to find ghost primes for (p, r), if you let b equal or exceed r, every integral y will have a factor of r, and every non-integral y is by definition not prime. You will therefore find no g for (p, r) if $b \geq r$.

4.5 Theorem: If b = (candidate g), z has a factor of (candidate g)

To search for the *smallest* g, you might start with a small primorial, and test candidates for g; plug in some prime ending $\{1, 3, 9\}$, and congruent to 5 (mod 6). From those, obtain a z (a candidate r), then check to see if both z and $z-2$ are prime. If they are, then $(z-2, z)$ is a prime pair, and the candidate g is a g. If either z or $z-2$ are composite, you might increment the primorial, and evaluate again. But how far out do you have to keep incrementing the primorial?

As in the above example, there exists an upper limit for the primorial:

Theorem 4. *Let g_c be some prime candidate for g, and let z be a candidate for r*

$$\text{For } z = \frac{b\#}{2} - 4g_c$$
$$\text{If } b \geq g_c, \text{ then } g_c \text{ is a factor of } b\#, \text{ and } g_c \text{ will divide } z.$$

Proof. Start with the equation for $2z$. Let $b = g_c$, and let f be the prime number (whatever it is) that comes before g_c.

$$2z = b\# - 8g_c$$

Expand $b\#$, and check for a factor: divide $2z$ by g_c

$$\frac{2z}{g_c} = \frac{(2 \cdot 3 \cdot 5 \cdot \ldots \cdot f \cdot g_c) - 8g_c}{g_c}$$
$$\frac{2z}{g_c} = (2 \cdot 3 \cdot 5 \cdot \ldots \cdot f) - 8$$
$$\frac{z}{g_c} = (3 \cdot 5 \cdot \ldots \cdot f) - 4$$
$$\frac{z}{g_c} = \frac{f\#}{2} - 4 \tag{6}$$

The right side of (6) is an integer for any prime f; if b equals g_c, then z has a factor of g_c.

If you let b exceed g_c, z will still have a factor of g_c

$$\frac{2z}{g_c} = \frac{(2 \cdot 3 \cdot 5 \cdot \ldots \cdot f \cdot g_c \cdot b) - 8g_c}{g_c}$$
$$\frac{2z}{g_c} = (2 \cdot 3 \cdot 5 \cdot \ldots \cdot f \cdot b) - 8$$
$$\frac{z}{g_c} = (3 \cdot 5 \cdot \ldots \cdot f \cdot b) - 4$$
$$\frac{z}{g_c} = \frac{b\#}{2g_c} - 4 \tag{7}$$

The right side of (7) is an integer for any prime $b > g_c$. Therefore, if $b \geq g_c$, then (since g_c is a factor of $b\#$), z has a factor of g_c □

You need increment only to the prime f, which is the prime just before the g_c you are checking. For example, if you are checking to see if 23 is one of the smallest g, how far do you have to keep incrementing the primorial? No further than 19#. If you increment to 23# or larger, here's what happens:

Example 2.

$$\frac{2z}{23} = \frac{(2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23) - 8\cdot 23}{23}$$
$$\frac{2z}{23} = (2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19) - 8$$
$$\frac{z}{23} = (3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19) - 4$$
$$\frac{z}{23} = \frac{19\#}{2} - 4 \tag{8}$$

The right side of (8) is an integer; z has a factor of 23. Keep $b < g_c$*, and z will not have a factor of* g_c*. For instance, let* $b = 7$*:*

$$\frac{2z}{23} = \frac{(2\cdot 3\cdot 5\cdot 7) - 8\cdot 23}{23}$$
$$\frac{2z}{23} = \frac{210 - 8\cdot 23}{23}$$
$$\frac{z}{23} = 0.56521\ldots$$

In the above example, z has no factor of 23, nor any positive factors except 1 and z

$$2z = (2\cdot 3\cdot 5\cdot 7) - 8\cdot 23$$
$$2z = 210 - 8\cdot 23$$
$$2z = 26$$
$$z = 13 \textit{ (which is an r)}$$

For finding and analyzing the number of the g, this is good news; it means that the number of g linked to any one r is not arbitrary, nor can it become infinite. There are only some finite number of g linked to any given r, no matter how large r becomes. No r is linked to infinitely many g.

My restrictions actually reflect the reality that there is no use searching for g, or for r in primorials $\geq r\#$. To obtain the full set of g you must obtain not just the first prime evaluation, but all the prime evaluations. To do that, you must check for primality all evaluations up to $\frac{p\#}{2}$ for each r.

4.6 A Reiteration of g

I am yet puzzled by the instance of a reiteration of g. Up until the prime pair $(1061, 1063)$ there were no reiterated g. But that pair repeated an earlier g

A Reiteration of g

pair	g	$b\#$
$(11, 13)$	23	$7\#$
$(1061, 1063)$	23	$11\#$

I am not yet certain what caused the reiteration. But the two linked sets, $\{13, 23, 7\}$ and $\{1063, 23, 11\}$ are still unique. Also, $r = 1063$ is part of at least two other unique linked sets – each with different g. Such reiterations do not adversely affect *Conjecture 2*. For the reciprocal sum of the g, however, I have made a rule: No repetitions of g are permitted. My rule may sound arbitrary, but the existence of such repetitions does not mean that, in the sequence of the g, there are duplicate g. Apparently, the same g may appear more than once if linked to a different r, and to a different half-primorial. I consider it correct to toss repetitions of g for the sum.

5 Conclusions

I have supplied empirical evidence that *Conjecture 2* is true. If it is true, then, for each (p, r), the p is in at least one unique linked set $\{p, g, b\}$, and the r is in at least one unique linked set $\{r, g, b\}$, where the g and the b are the same in both sets. If *Conjecture 2* is true, and if the g are infinitely many, then *Conjecture 3* is true.

I have found no exception to *Conjecture 2*, and it seems to me very improbable that an exception exists. But I have not proven *Conjecture 2*. And *Conjecture 3* cannot be proven until *Conjecture 2* is proven (or unless the TPC is proven in some other way).

There is an interesting mix of cases here (which are easily conflated):

Case 1: If each $r > 5$ is in a unique linked set $\{r, g, b\}$, and if the sequence of the g is infinite, its infinitude would mean the r would be infinitely many.

Case 2: If each $r > 5$ is in a unique linked set $\{r, g, b\}$, and if the sequence of the g is finite, its finiteness would mean the r would be finite.

Case 3: If some $r > 5$ is not in a unique linked set $\{r, g, b\}$, (there is no evidence that this is the case), and if the sequence of the g is infinite, its infinitude would tell us nothing about the infinitude of the r, or the lack thereof.

Case 4: If some $r > 5$ is not in a unique linked set $\{r, g, b\}$, and if the sequence of the g is finite, its finiteness would tell us nothing about the infinitude of the r, or the lack thereof.

I originally conflated the above, so that I considered just Case 1. So, I made only one conjecture, where two are needed (This was pointed out to me by Greathouse).

It also seemed to me at first that *Conjecture 2* would imply the TPC by considering only some prime pairs, but always finding larger ones. I observe, however, that *Conjecture 2* doesn't actually skip any prime pairs. If you use a large enough half-primorial, you can obtain any prime pair, although the smaller pairs require smaller half-primorials.

There seems to be no end to the unique g. I have so far found only one instance where a g is reiterated (I wrote about it here: 4.6). Even though at least one g is linked to more than one r, this does not indicate that the g might be finite; the sequence of unique g is more than twice as large as that of the r.

Specifically, there are 37 ghost primes linked to the prime pairs > 3 and ≤ 281. From those stats, I calculated what might be called the *ghost prime quotient*. To calculate it, you first need to look up or calculate $\pi_2(x)$, which is the number of prime pairs $\leq x$. I use $\pi_2(x) - 1$ because $(3, 5)$ is too small for a ghost prime to be linked to it; the linking of the ghosts to the twin primes begins with $(5, 7)$.

$$\begin{aligned} \pi_2(281) - 1 &= 18 \\ \frac{\text{the number of unique } g \text{ linked to the first } \pi_2(281) - 1 \text{ prime pairs}}{\pi_2(281) - 1} &= \frac{37}{18} \\ &= 2.0555\ldots \end{aligned}$$

I have not calculated *all* the ghosts for any pair beyond $(281, 283)$, because it's less work to calculate just the *first* ghost (subsequent ghosts get very large very quickly). So the present value of above quotient is exceeding rough. To make it more accurate, one would need to obtain much more data. But it appears that the *ghost prime quotient* increases with increasing $\pi_2(x)$.

Does knowledge of the g help prove the *Twin Prime Conjecture*? I don't think it hurts. It might be a new piece to the puzzle. It might help people see something about the twin primes they didn't see before.

If the TPC were false, I suppose the twin primes would (after some very large and unspecified twin) cease. The primes are infinite, so they would necessarily continue, except that no longer would any be found in the sequence $(p,\ p + 2)$. With knowledge of the g, must I amend that perception? Must I now say that, if the TPC were false, then not only would the $(p,\ p + 2)$ cease, but also the g?

Yes. The g would not continue as defined, since the g are by definition linked to the r, and there would be no more r. As defined, the g exist (and are more numerous than the r). If indeed each $r \geq 7$ is in a unique linked set $\{r, g, b\}$, then, if either the g sequence or the r sequence is finite, the other sequence is also finite.

I like to think about the q becoming extinct after some huge integer, and what that would imply for the sequence of the primes. After all, if someone can prove that the q either cease or don't cease, then the *Twin Prime Conjecture* would be settled. But there exists no empirical evidence (save the convergence of their reciprocal sums, and their thinning out with increasing x) that the prime pairs might cease. So I have not thought about how to redefine the g, were the q to become extinct, nor has that been the focus of this article.

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A Ghost Primes Are Not Like Chen Primes

Chen Primes are primes p, such that $p + 2$ is either prime, or is a product of 2 primes. Some ghost primes > 2 are primes g, such that $g + 18$ is either prime, or is a product of 2 primes:

Small ghost primes + 18, and factors

g	$g + 18$	factors	Chen-like?
2	20	$2 \cdot 2 \cdot 5$	tossed
23	41	prime	y
63809	63827	$83 \cdot 769$	y
281	299	$13 \cdot 23$	y
63803	63821	$19 \cdot 3359$	y
11	29	prime	y
1212443	1212461	$29 \cdot 41809$	y
263	281	prime	y
3719	3737	$37 \cdot 101$	y
251	269	prime	y

In 1973, Jing Run Chen proved that Chen Primes are infinite. If it were true that all ghost primes were like Chen Primes (using $p + 18$ instead of $p + 2$), then ghost primes might easily be proven infinite. My hopes were high until I checked a larger g:

$$402205595917584113487385569239 + 18 = 402205595917584113487385569257$$
$$402205595917584113487385569257 = 37 \cdot 38543 \cdot 282033612103003324112827$$

Not like Chen Primes for $p + 18$. I have not checked beyond $p + 18$.

B Only Some g Are Sophie Germain

Rearrange $\frac{b\#}{2} = 4g + r$ to:

$$\frac{b\#}{2} = 4g + 2 + p$$
$$\frac{b\#}{2} = 2(2g + 1) + p$$

In the above, the quantity $(2g + 1)$ is the form of the larger prime of a *Sophie Germain* couplet for the prime g. In some instances, g is in fact a Sophie Germain prime. But not in every instance. 263, for instance, is a g, but $(2 \cdot 263 + 1)$ is composite.

C A Very Simple Theorem

Theorem 5. *Each primorial* ≥ 30*, when divided by 2, will give a quotient that ends with a 5.*

Proof. Start with $5\# = 30$. To obtain each subsequent primorial, 30 must be multiplied by each subsequent prime, and all of them are odd. Whenever an odd integer (such as 3) is multiplied by an odd integer, the product will end with an odd digit. When 30 (or $3 \cdot 10$) is multiplied by an odd integer, the product will be ($3 \cdot$ an odd integer $\cdot 10$). The product will end with an odd digit followed by a zero. When that product is in turn multiplied by an odd integer, the new product will also end with an odd digit followed by a zero. And so on for all such products. Therefore, all subsequent primorials will be integers ending with an odd digit followed by a zero. But exactly which odd digits followed by a zero do the primorials end in?

No primorial ends in 50, because, to get an integer that ends in 5, you must multiply an integer ending in 5 by an integer ending in an odd digit. Similarly, to get an integer that ends in 50, you must either multiply an integer ending in 50 by an integer ending in an odd digit, or multiply an integer ending in 5 by an integer ending in an odd digit followed by a zero. If 5# ended 50, then all subsequent primorials would end 50. Happily, 5# ends 30, and no prime > 5 ends in 5. Primorials $\geq 5\#$ are never multiplied by an integer ending in 5 to get the next primorial. Therefore, no primorial ends in 50. All primorials $\geq 5\#$ end $\{10, 30, 70\, or\, 90\}$. Any of those divided by 2 gives an integer ending in 5. Return link: 2 □

D Copyright and References

References

[1] D. A. Goldston, *Are the Twin Primes Infinitely Many?* (2009), p. 2, www.math.sjsu.edu/, ↑1

[2] C. R. Greathouse IV, *PARI script* (2012) oeis.org/, ↑3.3

[3] C. R. Greathouse IV, *Table of n, a(n) for n = 1..204* (2012) oeis.org/, ↑3.3

[4] M. G. Kaarhus, *OEIS Sequence A218046* (2012) oeis.org/, ↑3.3